Lecture 9: Sheaves of $\mathcal{O}_X$-Modules, $X = \operatorname{Proj} S$

S = graded ring, M = graded S-module

$\rightsquigarrow \tilde{M}$ sheaf on $X = \operatorname{Proj}(S)$

$$\tilde{M}(U) = \left\{ s : U \longrightarrow \coprod_{\mathfrak{p} \in U} M_{(\mathfrak{p})} \;\middle|\; \begin{array}{l} \forall \mathfrak{p} \in U,\ \exists V_{\mathfrak{p}} \subseteq U \\ \text{s.t. } s(\mathfrak{q}) = \frac{m \in M}{f \notin \mathfrak{q}},\ \forall \mathfrak{q} \in V \\ \deg(m) = \deg(f) \end{array} \right\}$$

Proposition 1. ① $(\tilde{M})_{\mathfrak{p}} \cong M_{(\mathfrak{p})}$

② $f \in S_+$ homog, $\tilde{M}|_{D_+(f)} \cong (\widetilde{M_{(f)}})$

In particular, $\tilde{M}$ is (quasi-)coherent
if S Noetherian, M finitely generated

Definition: $S(n)$ graded S-module w/ $S(n)_d = S_{n+d}$

$\mathcal{O}_X(n) := \widetilde{S(n)}$ $\quad$ $\mathcal{O}_X(1)$ twisting sheaf of Serre

$\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$ for any sheaf of $\mathcal{O}_X$-module $\mathcal{F}$

Proposition 2: S is generated by S_1 as S_0-algebra

① $\mathcal{O}_X(n)$ invertible sheaf on X

② $\tilde{M}(n) \cong \widetilde{M(n)}$. $\quad \therefore \mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \cong \mathcal{O}_X(n+m)$

③ T: graded ring, generated by T_1 as T_0-algebra

$Y := \operatorname{Proj} T$

$S \xrightarrow{\varphi} T$ degree preserving

$\rightsquigarrow U = \{ \mathfrak{p} \in \mathrm{Proj}\, T \mid \mathfrak{p} \not\supseteq \varphi(S_+) \} \xrightarrow{f} X$

$f^* \mathcal{O}_X(n)|_U \cong \mathcal{O}_Y(n)|_U \quad , \quad f_*(\mathcal{O}_Y(n)|_U) \cong (f_* \mathcal{O}_U)(n)$

pf: ① Given $f \in S_1$, $\mathcal{O}_X(n)|_{D_+(f)} \cong \widetilde{S(n)_{(f)}}$

$$\begin{array}{ccc} S_{(f)} & \xrightarrow{\cong} & \underline{S(n)_{(f)}} \\ \cup & & \cup \\ s & \longmapsto & f^n s \end{array}$$

consists of elements in S_f w/ deg $= n$

$\because \deg(f) = 1$

$\therefore S(n)_{(f)}$ free $S_{(f)}$ – module

$\Rightarrow \mathcal{O}_X(n)$ free $\mathcal{O}_X$–module on $\underline{D_+(f)}$
form base of topology

② generally M, N graded S–module

$\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \cong (\widetilde{M \otimes_S N})$ $\quad M_{(f)} \otimes_{S_{(f)}} N_{(f)} \cong (M \otimes_S N)_{(f)}$

③ $S \xrightarrow{\varphi} T$ $\quad \rightsquigarrow \quad Y = \mathrm{Proj}\, T \supseteq U \longrightarrow X = \mathrm{Proj}\, S$

$\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$

$\varphi^{-1}(\mathfrak{p}) \not\supseteq S_+ \quad \mathfrak{p} \not\supseteq \varphi(S_+)$

generally, M: graded S-module. $\quad f^*(\widetilde{M}) \cong (\widetilde{M \otimes_S T})|_U$

N: graded T-module $\quad f_*(\widetilde{N}|_U) \cong (\widetilde{{}_S N})$

ex. locally free sheaf of $\mathcal{O}_X$-module $\longleftrightarrow$ vector bundle on X

$\mathcal{O}_X(-1)$ on $\mathbb{P}^n \longleftrightarrow$ tautological line bundle of $\mathbb{P}^n$

Lemma 1: X = Noetherian scheme. $\mathcal{L}$ = invertible sheaf on X

$f \in \Gamma(X, \mathcal{L})$, $X_f = \{x \in X \mid f_x \notin m_x \mathcal{L}_x\}$

$\mathcal{F}$ = quasi-coherent sheaf on X

① If $s \in \Gamma(X, \mathcal{F})$ s.t $s|_{X_f} = 0$,

then $f^n s = 0$, for some $n > 0$ injectivity

$\in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$

$\Gamma(X_f, \mathcal{F}) \xleftarrow{\cong} \Gamma(X, \mathcal{F})_f$

② Given $t \in \Gamma(X_f, \mathcal{F})$, then $\exists n > 0$

s.t $f^n t$ extends to X surjectivity

pf: This is a global version of Lemma 1 in Lecture 8.

① $X = \bigcup_i U_i$ finite affine open cover $\because X$ quasi-compact

$U_i \cong \operatorname{Spec} A_i$ s.t $\mathcal{L}|_{U_i}$ is free

i.e $\exists\ \mathcal{L}|_{U_i} \xrightarrow[\cong]{\psi_i} \mathcal{O}_X|_{U_i}$

X Noetherian + $\mathcal{F}$: quasi-coherent $\Longrightarrow$ $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$

$s \in \Gamma(X, \mathcal{F}) \longleftrightarrow m_i \in M_i$

$\Gamma(X, \mathcal{L}) \longrightarrow \Gamma(U_i, \mathcal{L}) \cong \Gamma(U_i, \mathcal{O}_{U_i})$ $X_f \cap U_i = D(\psi_i(f))$

$f \longmapsto \psi_i(f) \in A$

$s|_{X_f} = 0 \Rightarrow s|_{D(\psi_i(f))} = 0 \Rightarrow \psi_i(f)^{n_i} \cdot s = 0 \in M_i$, for some $n_i > 0$

Lemma 1 in Lecture 8 i.e $f^{n_i} s = 0$ on U_i only finitely many i

② Similar to Lemma 1. ② in Lecture 8

Just need $U_i \cap U_j$ can be covered by finitely many affine charts.

Proposition 3. S = graded ring, finitely generated by S_1 as S_0-algebra

$X = \mathrm{Proj}(S)$, $\mathcal{F}$: quasi-coherent on X

$$\Gamma_*(\mathcal{F}) := \bigoplus_{n \geq 0} \Gamma(X, \mathcal{F}(n)) \longleftarrow \text{geometric interpretation?}$$

$$\Longrightarrow \widetilde{\Gamma_*(\mathcal{F})} \cong \mathcal{F} \quad \text{canonically}$$

pf: We will prove the proposition on each affine chart

Notice that $\widetilde{\Gamma_*(\mathcal{F})}$ is quasi-coherent

observation: $\mathrm{Hom}_{A\text{-mod}}(M, \Gamma(X, \mathcal{F})) = \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F})$ for $X = \mathrm{Spec} A$

$$\varphi \longmapsto \tilde{\varphi}: \tilde{M}(D(f)) \longrightarrow \mathcal{F}(D(f)) \text{ is } \mathcal{O}_X(D(f))\text{-module} \cong A_f$$

$$\frac{m}{f^k} \longmapsto \varphi(m) \cdot \frac{1}{f^k}$$

$$\varphi \longleftarrow \tilde{\varphi}$$

taking global section

Thus, it suffices to understand

$$\beta: \widetilde{\Gamma_*(\mathcal{F})}(D_+(f)) \longrightarrow \mathcal{F}(D_+(f)), \quad \forall f \in S_1$$

$m \in \Gamma(D_+(f), \mathcal{F}(k))$

$f^k \in \Gamma(D_+(f), \mathcal{O}_X(-k))$

$$\frac{m}{f^k} \longmapsto m \otimes f^{-k}$$

$$\text{Lemma 1} \Rightarrow \mathcal{F}(D_+(f)) \underset{\beta}{\cong} \Gamma_*(\mathcal{F})_{(f)}, \quad \forall f \in S_1$$

S generated by S_1 as S_0-algebra

$\Longrightarrow$ X can be covered by $D_+(f)$, $f \in S_1$

ex. $X = \operatorname{Proj} \underbrace{A[x_1, \cdots, x_n]}_{\overset{\shortparallel}{S}}$, $\Gamma_*(\mathcal{O}_X) = \bigoplus_{d \geq 0} \Gamma(X, \mathcal{O}_X(d))$ $=?$

$t \in \Gamma(X, \mathcal{O}_X(d)) \Longleftrightarrow$ $t_i \in \Gamma(D_+(x_i), \mathcal{O}_X(d)) \cong S(d)_{(x_i)}$

i.e. homog. of deg. d in S_{x_i}

w/ $t_i \big|_{D_+(x_i x_j)} = t_j \big|_{D_+(x_i x_j)}$

x_i not zero divisor of S

$\Longrightarrow$ $S \hookrightarrow S_{x_i} \hookrightarrow S_{x_i x_j} \hookrightarrow \cdots \hookrightarrow S_{x_0 x_1 \cdots x_n}$

$\therefore$ $\Gamma_*(\mathcal{O}_X) = \bigcap_i S_{x_i} \subseteq S_{x_0 \cdots x_n}$

$\rotatebox{90}{$\in$}$

$t = x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n} \underline{f(x_0, \cdots, x_n)}$

homog. polynomial not divisible by x_i

$t \in S_{x_i}$ iff $i_j \geq 0$ for $j \neq i$

$\therefore$ $\bigcap_i S_{x_i} = S$

ex. $S = \mathbb{C}[x,y]/(x^2)$, then $\Gamma_*(\mathcal{O}_X) \not\cong S$

$X = \operatorname{Proj} S$

Proposition 4: A = ring

$Y \xhookrightarrow[\text{closed}]{} \mathbb{P}^n_A$, then $\exists\, I \triangleleft A[x_0, \cdots, x_n]$ homogeneous
we say Y is projective over A

s.t. $Y = \mathrm{Proj}(S/I)$

Moreover, $Y = \mathrm{Proj}\, S$, S graded ring w/ $S_0 = A$
finitely generated by S_1 as S_0-alg.

pf: Denote $\mathcal{I}_Y$ = sheaf of ideal of $Y \hookrightarrow X = \mathbb{P}^n_A$
(quasi-coherent)

$$\mathcal{I}_Y \hookrightarrow \mathcal{O}_X \Longrightarrow \Gamma_*(\mathcal{I}_Y) \hookrightarrow \Gamma_*(\mathcal{O}_X) \cong A[x_0, \cdots, x_n]$$

$\otimes$: exact

Γ : left exact

$$\Gamma_*(\mathcal{I}_Y) = \underbrace{\bigoplus_{n \geq 0} \Gamma(X, \mathcal{I}_Y(n))}_{\mathfrak{a}} \hookrightarrow \underbrace{\bigoplus \Gamma(X, \mathcal{O}_X(n))}_{S}$$

$\mathfrak{a} = \bigoplus_{d \geq 0} S_d \cap \mathfrak{a}$ thus $\Gamma_*(\mathcal{I}_Y)$ homog.

Conversely, given homog. ideal $\mathfrak{a} \triangleleft S$

$\mathrm{Proj}(S/\mathfrak{a}) \hookrightarrow \mathrm{Proj}(S)$ w/ ideal sheaf $\tilde{\mathfrak{a}} = \widetilde{\Gamma_*(\mathcal{I}_Y)} \underset{\text{Proposition 3}}{\cong} \mathcal{I}_Y$

Thus, $Y \cong \mathrm{Proj}(S/\mathfrak{a})$

the later part follows from the analogue statement of S.

Definition: ① Y scheme, $\mathcal{O}(1)$ on $\mathbb{P}^n_Y := \mathbb{P}^n_{\mathbb{Z}} \times_{\mathrm{Spec}\,\mathbb{Z}} Y$ is defined to be the pull back of $\mathcal{O}(1)$ on $\mathbb{P}^n_{\mathbb{Z}}$

② $X \longrightarrow Y$, $\mathcal{L}$ invertible on X

$\mathcal{L}$ is <u>very ample</u> if $\exists$

$$\begin{tikzcd} X \arrow[r, hook, "i", "\text{closed immersion}"'] \arrow[dr] & \mathbb{P}^n_Y \arrow[d] \\ & Y \end{tikzcd}$$

. for some n

s.t. $\mathcal{L} = i^*\mathcal{O}(1)$

Definition: X scheme. $\mathcal{F}$: sheaf of $\mathcal{O}_X$-module

$\mathcal{F}$ is <u>generated by global sections</u>

if $\exists\ s_1, \cdots, s_k \in \Gamma(X, \mathcal{F})$ s.t. $s_{1,x}, \cdots, s_{k,x}$ generates $\mathcal{F}_x$ as $\mathcal{O}_x$-module, $\forall x \in X$

ex. Any quasi-coherent sheaf on an affine scheme is generated by global sections.

ex. $X = \operatorname{Proj} S$, S graded ring, finitely generated by S_1 as S_0-algebra

then $\mathcal{O}(1)$ is generated by global sections

$S_1 \subseteq \Gamma(X, \mathcal{O}(1))$

ex. $f: X \to Y$, $\mathcal{G}$ generated by global sections

then $f^*\mathcal{G}$ is generated by global sections.

$m_1, \cdots, m_k \in M$, generate M as A-module

$\Rightarrow m_1 \otimes 1, \cdots, m_k \otimes 1 \in M \otimes_A B$ generate $M \otimes_A B$ as B-module ex. If $\mathcal{F}, \mathcal{G}$ generated by global sections, then so is $\mathcal{F} \otimes \mathcal{G}$.

ex. $\mathcal{F}$ = invertible sheaf on X

$\mathcal{F}$ is generated by global sections iff $\forall x \in X$

$\exists s \in \Gamma(X, \mathcal{F})$ s.t. $s_x \notin m_x$

Nakayama's lemma: $\mathfrak{a} \triangleleft A$, M: finitely generated A-module

If $\mathfrak{a}M = M$, then $\exists f \in 1+\mathfrak{a}$ st $fM = 0$

Corollary: A: local ring M: A-module "Implicit function theorem"

$m_1, \cdots, m_k$ generate M as A-module

iff $m_1, \cdots, m_k$ generate M/mM as A/m-vector space

pf: $\mathfrak{a} = m$, then elements in $1+\mathfrak{a}$ are invertible

Apply Nakayama's lemma for A-module $M / \oplus m_i A = 0$

Theorem: $\mathcal{F}$ = coherent sheaf on X scheme

$\varphi(x) := \dim_{k(x)} \mathcal{F}_x \otimes k(x)$, $\forall x \in X$

is upper-semicontinuous

pf: Assume that $\mathcal{O}^{\oplus n} \xrightarrow{\varphi} \mathcal{F}$ surjective at p

then apply Nakayama's lemma to $\underline{\mathcal{F}/\mathrm{Im}\,\varphi}$

coherent w/ closed support

Theorem (Serre) X projective scheme over a Noetherian ring A

$\mathcal{O}_X(1)$: very ample on X

$\mathcal{F}$: coherent sheaf on X

Then $\exists\, n_0 \in \mathbb{N}$ s.t $\mathcal{F}(n)$ is generated by global sections

for all $n \geq n_0$.

pf: $X \underset{\text{closed immersion}}{\overset{i}{\hookrightarrow}} \mathbb{P}^n_A$, $\mathcal{O}_X(1) = i^* \mathcal{O}(1)$

$\xrightarrow[\text{Proposition 2}]{} \quad i_*(\mathcal{F}(n)) \cong (i_* \mathcal{F})(n)$

Since $i_*\mathcal{F}(n)$ & $\mathcal{F}(n)$ have the same global sections

$\mathcal{F}(n)_x = i_*(\mathcal{F}(n))_{i(x)}$

It suffices to show the theorem for $X = \mathbb{P}^n_A$

Now cover X w/ $D_+(x_i)$

$\mathcal{F}|_{D_+(x_i)} \cong \widetilde{M_i}$, M_i finitely generated $A\left[\frac{x_0}{x_i}, \cdots \widehat{\frac{x_i}{x_i}}, \cdots, \frac{x_n}{x_i}\right]$-module

say $s_{ij} \in M_i$ generaters

$\qquad\qquad\quad \cong$

$\Gamma(D_+(x_i), \mathcal{F})$

replaced by $n = \max\{n_{ij}\}$

Proposition 2 ② $\Longrightarrow$ $x_i^{n_{ij}} s_{ij} \in \Gamma(X, \mathcal{F}(n))$

then $x_i^n s_{ij}$ generate $\mathcal{F}(n)$ on $\bigcup_i D_+(x_i) = X$

Corollary: X = projective scheme over a Noetherian ring A.
$\mathcal{F}$ = coherent sheaf on X

$\Longrightarrow$ $\mathcal{F}$ is a quotient of $\bigoplus_{i=1}^{n} \mathcal{O}(n_i)$, for some n_i, n.

pf: $\mathcal{F}(n)$ is generated by global sections, for $n \gg 0$
say $s_1, \cdots, s_k$

then $\underbrace{\bigoplus_{i=1}^{k} \mathcal{O}_X \cdot s_i}_{\cong \bigoplus_{i=1}^{k} \mathcal{O}_X} \longrightarrow\!\!\!\!\rightarrow \mathcal{F}(n) \longrightarrow 0$

$\otimes \mathcal{O}_X(-n)$ exact functor

$\bigoplus_{i=1}^{k} \mathcal{O}_X(-n) \longrightarrow\!\!\!\!\rightarrow \mathcal{F}$

Theorem: X = projective scheme over Noetherian ring A
$\mathcal{F}$ = coherent sheaf on X

$\Longrightarrow$ $\Gamma(X, \mathcal{F})$ finitely generated A-module

Actually true for all cohomology $H^i(X, \mathcal{F})$